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Motion Out of Noisy States

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The relative occupation of competing states of local stability is not determined solely by the characteristics of the locally favored states, but depends on the noise along the whole path connecting the competing states. This is not new, but the sophistication of most modern treatments has obscured the simplicity of this central point, and here it is argued for in simple physical terms. In addition, recent work by van Kampen and by Büttiker, for particles in closed loops, subject to a force field, heavy damping, and a temperature which is a function of position in the loop, are supplemented. In that case, circulating currents are set up, and these are evaluated. A final speculative section emphasizes the difficulty in calculating the long-term time evolution of the probability distribution in complex multistable systems with state-dependent noise.

KEY WORDS: Relative stability; circulating currents; temperature gradients; force fields; state-dependent noise.

1. INTRODUCTION

Particles in a system with nonuniform temperature move out of the hot regions with greater velocity than out of the old regions. Pebbles in a driveway on flat land accumulate on the side (this example is due to G. E. Hinton). In the driveway they are agitated (hot region). They are left undisturbed on the side (cold region), and therefore accumulate on the side. This occurs despite the fact that the traffic does not exert a directional force on the pebbles. The perfect gas law, $pV = nRT$, tells us that in the presence of a nonuniform temperature, but if we allow pressure equilibration, the density must vary as $1/T$. Nonuniform temperatures occur in a number of transport problems: thermoelectric effects, thermal diffusion and separation of isotopes,[1] thermophoresis,[2,3] etc. The method of equilibration varies; some of these problems allow equilibration via

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collisions, i.e., via pressure.[4] In the electrical case Coulomb interactions are allowed. Here I will emphasize the case of independent particles which do not interact with each other. These are presumed to be particles of the sort envisioned in the Smoluchowski equation. They can be subject to applied forces, and are coupled to a reservoir which determines a viscosity and also determines fluctuations. The reservoir temperature will be taken to be a function of position.

The detailed considerations will be one dimensional, i.e., the particles are located by specifying a single coordinate. The relevance, however, to higher dimensional problems is illustrated in Fig. 1. This shows a set of two-dimensional potential contours, with minima at A and B and a saddle point at S. In thermal equilibrium, the Boltzmann distribution $\rho \sim \exp(-U/kT)$ will prevail, and there is no particle transport. Now consider, instead, the case where the noise is not thermal equilibrium noise, and varies with position. Then, escape from the minima at A and B need not be via the saddle point if the noise activation for escape is much more pronounced along other paths. This permits the situation suggested in Fig. 1. Escape from A is via one path, that from B via another. In the steady state, where the two escape rates balance, we have a circulating

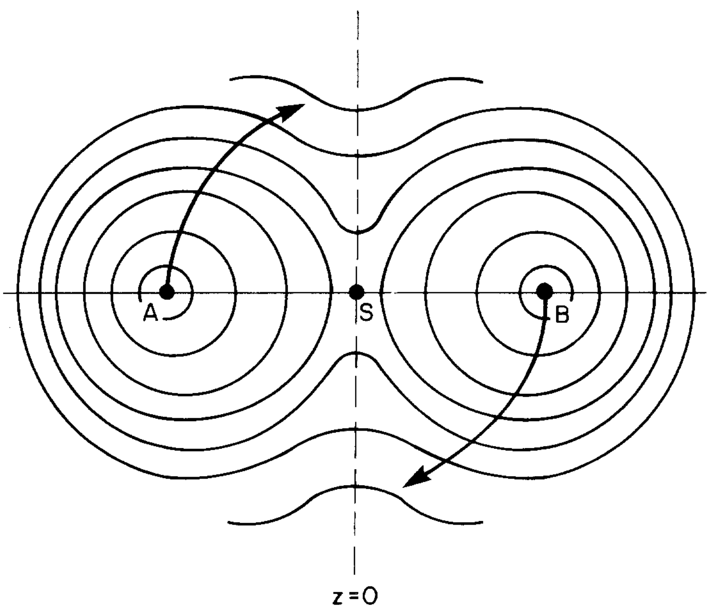

Fig. 1. Contours of a bistable potential with minima at A and B and a saddle point at S. Escape from the states of local stability can be along paths favored by particularly intense noise and need not go preferentially across the saddle point.

current.[5] The subsequent *one-dimensional* considerations will model noise-induced circulation of this sort around a loop.

The work reported here has a number of separate sources. The first step was implicit in ref. 6, made explicit in refs. 7 and 8, and elaborated upon on a number of later occasions. The point is that the relative likelihood of finding the system near competing states of local stability (e.g., potential minima) cannot be ascertained by examining the neighborhood of the competing states of stability. The noise behavior in the intervening unlikely states enters critically into the determination of relative stability. This point about the inadequacy of local criteria served to demonstrate the limited applicability of theorems which only examine entropy, entropy production, excess entropy production, etc., near the competing states of stability. Work based on such limited criteria has even been credited with explaining the origin of life.[9]

My later elaborations of refs. 7 and 8 did not focus on conditions which lead to circulation, as illustrated in Fig. 1. Such studies came out of the work of van Kampen[10,11] and Büttiker.[12] (Only refs. 10 and 12 deal explicitly with circulatory currents, but ref. 11 is intimately related, in turn, to ref. 10.) The relationship between this present paper and ref. 10 is particularly close. Papers dealing with noise which depends on the state of the system are, of course, abundant and often use the label multiplicative noise.[2] Most of this literature is, however, relatively far from the simple physical points we try to make here. For a recent general review see ref. 7.

2. NONUNIFORM TEMPERATURES

Consider the situation shown in Fig. 2, showing a double-well potential. In thermal equilibrium the density in the right-hand valley will be higher. Now introduce a hot zone, as shown in Fig. 3, in which the fluctuations are more intense. This aids escape from the right-hand well, but not from the left-hand well. In the steady state, the net exchange between the two wells must vanish; as a result, the hot zone will deplete the right-hand well population. With a suitable section of parameters, it becomes the less likely well.

For the squeamish, as well as for later use, I present a more careful argument. Assume that we are dealing with heavily damped motion, and that the particle, after crossing a temperature discontinuity, takes on its new temperature almost immediately. One possible physical embodiment for our situation, which may help the reader, is to consider this as a particle in a tube,[18] as shown in Fig. 4. When the particle, bounces into the

[2] See refs. 13–16 and the papers cited in ref. 14, particularly the list in ref. 12 of that paper.

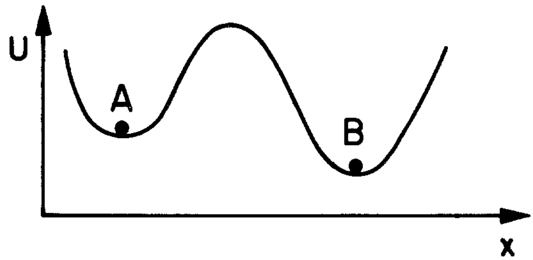

Fig. 2. Potential with two states of local stability. In equilibrium the lower-lying right-hand state will be more likely.

tube wall it is assumed to take on the temperature of the tube at that point. Reflection at the tube wall is furthermore assumed to be diffuse and not specular; the particle is reflected with equal probability into all angles. The particle can be taken to be charged and force fields maintained by charges on the outside of the tube. The heavily damped nature of the motion is reflected by the fact that the potential varies slowly compared to the tube diameter.

In view of the slow potential variation we can ignore it when considering the matching problem at a temperature discontinuity. (Even if the potential variation were not slow, we could still approximate the potential profile by a potential staircase, and similarly for the temperature. We could then choose the temperature steps to be between the potential

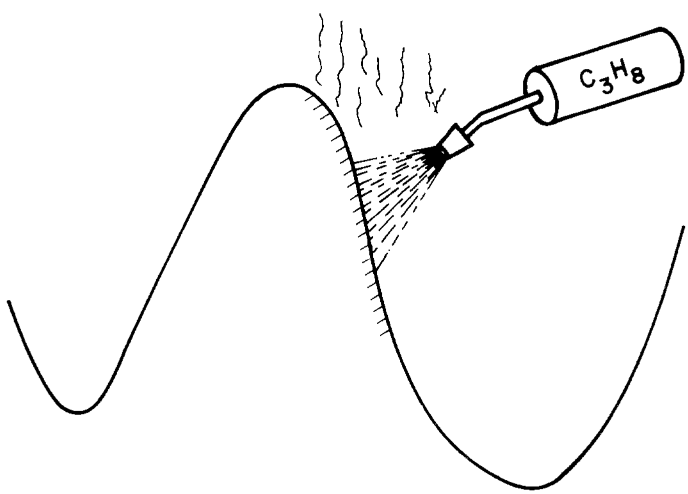

Fig. 3. Potential of Fig. 2 in which a portion of the right-hand well has been elevated to a higher temperature.

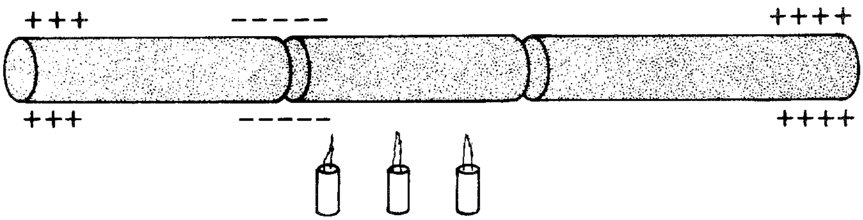

Fig. 4. Insulating tube containing a charged particle, with force fields maintained by charges on the outside of the tube. Different portions of the tube can be maintained at differing temperatures.

steps.) Figure 5 illustrates the temperature discontinuity. In a simple-minded approach, assume that carriers from the left with density ρ_1 and velocity v_1 bring a current $\rho_1 v_1$ to the discontinuity, balanced by a current $\rho_2 v_2$ from the right. Thus, $\rho_1 v_1 = \rho_2 v_2$. Utilizing $v \sim \sqrt{T}$, we find

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_1}{T_2}\right)^{1/2} \tag{2.1}$$

If, however, we are careful, we must admit that this simple argument does not do justice to the kinetics in the actual transition region shown in Fig. 5. After all, there is no single transverse plane along our model tube where the carriers from the left are entirely characteristic of the medium on the left whereas the carriers incident from the right are entirely characteristic of that medium. This is, however, a problem which is linear in the distribution function; we assume no interactions between particles. Therefore, even if Eq. (2.1) is incorrect, we can expect

$$\rho_2/\rho_1 = f(T_1, T_2) \tag{2.2}$$

with the right-hand side independent of the actual densities. Indeed, whether the specific form (2.1) is correct or a different answer allowed by Eq. (2.2) applies depends on model details. Thus, ref. 10 advocates $\rho \sim 1/T$ at a temperature discontinuity instead of Eq. (2.1).

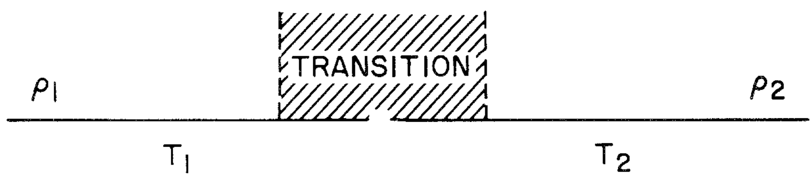

Fig. 5. Temperature discontinuity, with $T = T_1$ on the left and $T = T_2$ on the right. The region labeled transition indicates symbolically a range in which the transition in density from ρ_1 to ρ_2 is made.

I can illustrate the sensitivity to model details via a simple elaboration on the "tube" model. Consider, first, the original tube with no force field or temperature gradient. A spatially uniform density will be found in equilibrium. Now elevate a section of the tube to a higher temperature. The assumption of completely diffuse reflection at the tube wall assures us that the geometrical paths taken by the particles will not be affected by the temperature change. The temperature change only causes the velocity along these paths to be altered. Thus, in the presence of a nonuniform temperature, we immediately have $\rho \sim 1/\sqrt{T}$.

Now consider instead a variation on this case in which the tube expands with rising temperature. We now need to first consider an auxiliary case: A tube with the *geometry* present after establishing a temperature discontinuity but at thermal equilibrium at the original lower temperature. This is the temperature which, in the original problem, is presumed to be found in the left-hand portion of the tube. Now the auxiliary two-dimensional thermal equilibrium case will have a uniform volume density but a greater linear density on the right-hand side after integration over the cross section of the tube. Consider the transition from this auxiliary model to the case with the same geometry but with the nonuniform temperature. As a result of this latest change, the particle paths will be unaffected; only the velocities along these paths will change on the right-hand side. Thus, the volume densities on the right-hand side will diminish relative to those on the left by $(T_1/T_2)^{1/2}$, as in the original tube with unchanged cross section. After integrating over tube cross section, however, and considering the resulting one-dimensional projected density along the tube axis, the $(T_1/T_2)^{1/2}$ factor resulting from velocity changes will be offset by the larger volume available on the right-hand side. In fact, depending on the particular thermal expansion law, the geometrical effects may outweigh the velocity changes and cause the sign of the effect to be changed. The original tube model, with a geometry independent of temprature, corresponds to assuming a mean-free path independent of T. Instead, ref. 10 is based on the assumption that γ, the momentum relaxation rate, is temperature independent.

Thus, Eq. (2.1) is a specific result, valid for the temperature-independent cross section. Depending on model details, other results permitted by Eq. (2.2) can occur. Also of relevance here is a point made by N. G. van Kampen (personal communication) in connection with the type of tube discussed here, with a mean free path determined by collisions with the walls. Particles traveling almost straight down the axis have an exceptionally long mean free path. In the case of a two-dimensional tube, with a one-dimensional cross section, this leads to a divergence in the mobility. It is a logarithmic divergence and, as is typical in such cases, is removed by almost any artifice, e.g., a wiggle in the tube or baffles in the tube. Alternatively, the reader may want to restrict attention to a tube in three dimensions.

3. RELATIVE STABILITY CONTROLLED BY TEMPERATURE VARIATION

Consider a bistable well, of the type illustrated in Fig. 6. Between A and B as well as between C and D one is at the original temperature T_L. Between B and C a higher temperature T_H is maintained. Within each of the three regions AB, BC, and CD the Boltzmann distribution $\exp(-U/kT)$, with the applicable local temperature, holds in the steady state. At the temperature discontinuities Eq. (2.2) applies. In the equations that follow, B_+ and B_- represent points, respectively, just to the right or just to the left of the discontinuity at B, and similarly for C_+ and C_-. We thus find

$$\rho(B_-)/\rho(A) = \exp[-(U_B - U_A)/kT_L] \tag{3.1a}$$

$$\rho(B_+)/\rho(B_-) = f(T_L, T_H) \tag{3.1b}$$

$$\rho(C_-)/\rho(B_+) = \exp[-(U_C - U_B)/kT_H] \tag{3.1c}$$

$$\rho(C_+)/\rho(C_-) = 1/f(T_L, T_H) \tag{3.1d}$$

$$\rho(D)/\rho(C_+) = \exp[-(U_D - U_C)/kT_L] \tag{3.1e}$$

Multiplying all the ratios in Eq. (3.1) causes the ratios in (3.1b) and (3.1d) to cancel, yielding

$$\frac{\rho(D)}{\rho(A)} = \exp\left(-\frac{U_B - U_A}{kT_L}\right) \exp\left(-\frac{U_C - U_B}{kT_H}\right) \exp\left(-\frac{U_D - U_C}{kT_L}\right) \tag{3.2}$$

or

$$\frac{\rho(D)}{\rho(A)} = \exp\left(-\frac{U_D - U_A}{kT_L}\right) \exp\left[-\Delta U\left(\frac{1}{kT_H} - \frac{1}{kT_L}\right)\right] \tag{3.3}$$

where $\Delta U = U_C - U_B$. Thus, if BC is on the uphill side of the right well, as shown in Fig. 6, the heating increases the escape rate from the right well and the steady-state population of the left well. If the parameters are chosen correctly, then the lower well at D can be made the less likely state, via the temperature elevation in BC. This is the conclusion invoked in a number of my discussions: Relative stability cannot be determined by an examination of the neighborhoods of the favored states, such as A and B.

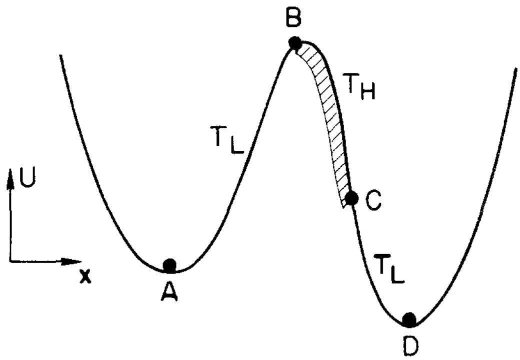

Fig. 6. Region between B and C has been evaluated to temperature T_H, with the remainder at the original temperature T_L.

The kinetics along the whole pathway, including the noise in the intervening rarely occupied states, must be considered. Thus, thermodynamic-type criteria, stressing the behavior near the local states of stability, cannot help.

4. CLOSED LOOP, WITHOUT CURRENT FLOW

Let us now make the transition to the case considered in refs. 10, and 12, where we still concentrate on motion along one spatial variable, but do so along a closed loop, so that temperature, potential, and distribution function (or density) ρ are all periodic. The Smoluchowski equation for our problem is

$$j = -\mu\rho \frac{dU}{dx} - kT\mu \frac{d\rho}{dx} - \alpha\mu k\rho \frac{dT}{dx} \tag{4.1}$$

Here U is the potential along the loop. The mobility μ can be temperature dependent. For simplicity, assume that there is no direct and purely spatial dependence of μ unrelated to temperature variations. Such a variation would not cause deviations from the thermal equilibrium distribution, $\rho \sim \exp(-U/kT)$. The first term on the rhs in Eq. (4.1) is the force-induced particle drift, the second is the usual diffusion term. These are the two terms which would be present at constant T. The final term allows for the effects discussed in the earlier sections; particles move out of the high-temperature regions more effectively than out of an adjacent low-temperature region. Thus, even if ρ is initially space independent and no force is present, a current can be expected in the presence of a temperature gradient. The undetermined and dimensionless constant α reflects the model dependence stressed in Section 2. I do, however, *assume* that α is space and temperature independent. Equation (4.1) allows for the possible dependence of the current on all the obvious variables via the terms proportional to dU/dx, $d\rho/dx$, and dT/dx. We know that if $dT/dx = 0$, the terms in dU/dx and $d\rho/dx$ must be related via the Einstein relation. Equation (4.1) has taken the diffusion current to be $D\, d\rho/dx$ rather than $d(D\rho)/dx$. This question has been discussed in detail elsewhere,[10,11,13,18,19] and will not be taken up here. Equation (4.1) can be mapped into Eq. (2.1) of ref. 12 by lumping the first and last terms of Eq. (4.1) into one effective drift term. Now, first consider Eq. (4.1) for the stationary state, if $dU/dx = 0$. In that case, Eq. (1), assuming $j = 0$, yields

$$\rho \sim (1/T)^{\alpha} \tag{4.2}$$

To get results in agreement with the tube model of Section 2 we have to choose $\alpha = 1/2$. For agreement with Eq. (3) of ref. 10 we have to choose $\alpha = 1$. For arbitrary U, Eq. (4.1), assuming $j = 0$, yields

$$\rho = \left(\frac{1}{T}\right)^{\alpha} \exp\left(-\int \frac{dU}{kT(x)}\right) \tag{4.3}$$

In the loop this has to be a periodic function of x, the coordinate around the loop. The prefactor $(1/T)^{\alpha}$ is periodic. The exponential term is periodic if

$$\oint \frac{dU}{T(x)} = 0 \tag{4.4}$$

This condition will be satisfied if the spatial variations of U and T are symmetrical about their maxima and minima and are in phase, in accordance with the predictions of ref. 12. Equation (4.4) is a condition already given in ref. 10 for the case of $\alpha = 1$, and generalized here to arbitrary α. As already stated, Eq. (4.1) can be mapped into the simpler form used in ref. 12 by lumping the terms in dU/dx and dT/dx into one drift term. In that terminology Eq. (4.4) becomes equivalent to Büttiker's condition that $\psi = \int (v/D)\, dx$ is a *periodic* function in the loop. Let us attempt to connect Eq. (4.4) with the entropy exchanged with the reservoir, by a particle traversing the whole loop. Note that particles passing through a short section of the loop do so by absorbing

$$dQ = dU + d(kT) \tag{4.5}$$

In the traversal around the whole loop, the entropy taken by the particle from the reservoir might plausibly be taken to be

$$\oint dS = \oint \frac{dQ}{T} = \oint \frac{dU}{T} + k \oint \frac{dT}{T} \tag{4.6}$$

The final term in (4.6) vanishes, and thus

$$\oint dS = \oint \frac{dU}{T(x)} \tag{4.7}$$

which must vanish if Eq. (4.4) is to be satisfied. Should $\oint dS = 0$ have been immediately apparent as the condition for no current flow? After all, the variation in T is not necessarily small, and we may be describing a system far from equilibrium. Even without a particle current around the loop the particles are transporting heat from the hot regions of the loop to the cold regions, and there must be a net entropy production. I believe, in fact, that $\oint dS = 0$ is *not* the correct equivalent of Eq. (4.4), and that the apparently appealing conclusion $\oint dS = 0$ is wrong. A particle moving from a higher temperature to a lower one deposits its energy at the final temperature, and not at some intermediate temperature, as implied in Eq. (4.6). Consider Eq. (4.6) applied to particles moving from position x_{i+1} at temperature T_{i+1} to x_i at T_i and in equal numbers in the reverse direction. The approach of Eq. (4.6) would yield no net entropy exchange. But clearly there is heat flow, and more entropy delivered to the reservoir at the low temperature than removed at the high temperature.

5. CURRENT FLOW

In this section I evaluate the current that flows in a loop not satisfying Eq. (4.4). For the sake of simple equations, I will use perturbation theory, assuming that second-order terms in the deviation from thermal equilibrium can be ignored. This approximation will give a physically more transparent result than the one yielded by the more accurate analysis of ref. 10. Within the perturbation theory approximation one can integrate Eq. (4.1) directly. I allow for the deviations in ρ from the equilibrium density and for the deviations in T from the original temperature but ignore cross products in these deviations. I assume a constant j and invoke periodicity. Rather than executing this in detail, I will follow a more physical procedure, yielding the same answer.

Let us assume a potential variation and temperature profile as shown in Fig. 7. Consider first the open loop situation discussed in Section 3, so

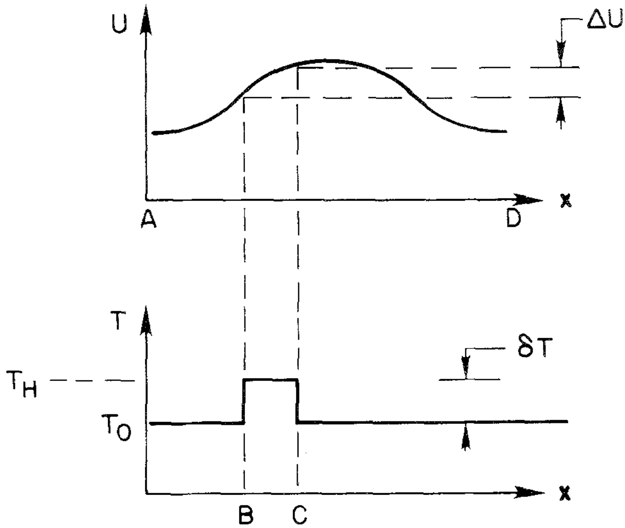

Fig. 7. Top: potential variation as a function of position x along a loop. Bottom: temperature variation, with an elevated temperature in BC.

that points A and D in Fig. 7 are unconnected except by passage over the intervening potential barrier. Equation (3.3) tells us that

$$\frac{\rho(D)}{\rho(A)}=\exp\left[-\Delta U\left(\frac{1}{kT_H}-\frac{1}{kT_0}\right)\right] \tag{5.1}$$

Within the perturbation theory approximation, assume a temperature difference

$$\delta T=T_H-T_0\ll T_0 \tag{5.2}$$

This permits us to write Eq. (5.1) in the form

$$\frac{\rho(D)}{\rho(A)}=\exp\left(\frac{\Delta U}{kT_0}\frac{\delta T}{T_0}\right) \tag{5.3}$$

or equivalently

$$\log\frac{\rho(D)}{\rho(A)}=\frac{\Delta U}{kT_0}\frac{\delta T}{T_0} \tag{5.4}$$

Now this open loop condition, with zero current, can be considered to be a superposition of two counteracting perturbations, each of which would cause a particle flow. These are:

I. Take $\rho(D) \neq \rho(A)$, but assume $T = T_0$ all along the device, in accordance with the perturbation assumption, which neglects cross products. As particles move out of the more heavily occupied valley, they are replaced; as they appear in the other valley, they are taken away. Thus, we maintain a steady-state current.

II. Assume $\delta T > 0$ as shown in the lower part of Fig. 7. But enforce $\rho(D) = \rho(A)$ by taking the particles arriving at D and returning them to A, effectively closing the loop.

Case II specifies the situation in which we are actually trying to evaluate a current, and we shall do so by evaluating the compensating current in I. Now the currentless situation characterized by Eq. (5.4) is not a strict superposition of I and II; after all, in I we have taken $T = T_0$. But as a result of the perturbation assumption we can neglect second-order cross influences.

What is the current flow in case I? If $\rho(D) > \rho(A)$, there is an electrochemical potential difference between the two valleys (at A and at D) equal to $kT_0 \log[\rho(D)/\rho(A)]$. We can regard this as a driving force applied to a medium with variable resistance, having good conductivity in the potential valleys and poor conductivity at the peak. This approach was introduced by Christiansen,[20] and has also been applied widely to the barrier crossing problem in the semiconductor device literature. It has been stressed in ref. 21 (see also Table I of ref. 16), but seems to have received little appreciation in the modern statistical mechanics literature. The resulting flux is

$$j_{\rm I} = KT_0 \log \frac{\rho(D)}{\rho(A)} \Big/ \int \frac{dx}{\rho_0 \mu} \tag{5.5}$$

The subscript I in Eq. (5.5) refers to situation I. The subscript in ρ_0 emphasizes that this is the unperturbed equilibrium distribution at T_0. Equation (5.5) is equivalent to Eq. (3.1) of ref. 12. The relationship between $\rho(D)/\rho(A)$ of situation I and δT of situation II is given by Eq. (5.4). Thus $j_{\rm II}$, which has the same magnitude as $j_{\rm I}$, is given by

$$j_{\rm II} = \Delta U \frac{\delta T}{T_0} \Big/ \int \frac{dx}{\rho_0 \mu} \tag{5.6}$$

Equation (5.6) can easily be generalized to situations in which there is a continuous variation of T, rather than the simple temperature profile of Fig. 7. This gives us

$$j = \left(\int dU \frac{\delta T}{T_0} \right) \Big/ \int \frac{dx}{\rho_0 \mu} \tag{5.7}$$

In Section 4 we learned that Eq. (4.4) gives the condition for $j=0$. Is this consistent with Eq. (5.7)? Within the perturbation approximation we can rewrite condition (4.4) as

$$\oint \frac{dU}{T} = \oint \frac{dU}{T_0 + \delta T} = \oint \frac{dU}{T_0} - \oint \frac{dU\, \delta T}{T_0^2} = 0 \tag{5.8}$$

$\oint dU/T_0$ vanishes. Thus, the condition in Eq. (5.8) becomes

$$\frac{1}{T_0} \oint \frac{dU\, \delta T}{T_0} = 0 \tag{5.9}$$

In that case, however, the current specified by Eq. (5.7) vanishes, demonstrating the consistency of Eq. (5.7) with Eq. (4.4).

6. COMPLEXITY

Section 3 emphasized that there are no short cuts in the calculation of relative stability; the detailed kinetics along the transition path must be taken into account. The relevance of this to discussions of evolution and the origin of life has been emphasized elsewhere.[16,18] In that case we are, of course, not dealing with two competing states of local stability, but a myriad of potentially stable ecologies. Mutations are the most obvious fluctuations which can induce transitions from one locally stable state to another one.

The lack of available short cuts in predicting the noise-activated exploration of many competing states of local stability can be related to questions of complexity in deterministic nonlinear systems, frequently treated in the literature. In discussing this relationship I will be repeating notions proposed in ref. 22 and will, admittedly, be entering into speculation. First consider chaos in deterministic systems. Multiply periodic motion, e.g., $y = \cos[\omega_0 t + (\delta \sin \omega_m t) t]$, or, for that matter, almost all functions expressed in typical mathematical notation, may look a little complex, but that does not cause unusual difficulty in the calculation of y at a much later time. We do not need to follow the motion in detail over the intervening period. Chaos is more complex than that, in a genuine way. To predict a position a great many "cycles" later in the chaotic case requires that we follow the motion very precisely. The program that directs such a calculation need not be complex; it is not a matter of great algorithmic complexity.[23] It is rather the detailed execution of the calculation which is long, i.e., the number of steps involved. This is a complexity measure which Bennett has proposed, discussed, and called "logical depth."[24] Bennett's "logical depth" is an attempt to go past the many obvious and superficial attempts to characterize self-organization as seen in biological structures. Bennett as well as Kuhn[25] have taught us in very different ways and very different language that there are some structures in nature (or computational results) which simply canot be reached easily from a simple or prevalent initial state. These states require a long evolution (or computation). That may well be the essence (or at least a part of it) of what we mean by self-organization. It has little to do with the almost instantaneous establishment of orderly motion in very simple deterministic systems, e.g., the Bénard instability, the Belousov–Zhabotinskii reaction, or the onset of oscillations in a self-excited oscillator, when the gain becomes large enough. Such simple systems are only following their laws of motion, and do so *immediately* when the external parameters are set appropriately. There is no long search or evolution involved in these simple instabilities.

The noise-activated search for likely states of local stability presents us with a similar dichotomy. If we are dealing with a multistable potential and thermal equilibrium noise, then the depth of a set of wells allows us to compute their eventual (long-term) relative probability densities. Even if we are dealing with systems of unlimited extent in which continually new parts of space will be occupied, the *relative occupation* of *nearby* states of local stability will not change appreciably after an initial period. If, however, we are dealing with a system which is not in thermal equilibrium, then it becomes more difficult. The additional difficulty relates to the discussion of the kinetics of Fig. 3, but also to the fact that in this more general case we can have circulation present in the steady state, as illustrated in Fig. 1. This opens, for example, the possibility of long circulation paths which can control the relative population of two nearby states but become fully effective only after a considerable time. The kinetics along the various pathways must then be taken into account. To predict the probability distribution at a much later time from a given initial state, we *must* follow the motion in detail. (The italicized word *must* in the preceding sentence is an intuitive assesment, it has not been proven.) A related measure of complexity, counting the number of states of local stability which have to be explored along the way, has been proposed by Kuhn.[25] He has called it *knowledge gained,* in connection with his discussions of the origin of life and the time development of biological evolution. In general, we can expect that there will be situations in which both ordinary chaos as well as our statistical complexities are present simultaneously and are not separable. We can start with a deterministic law of system motion which is already chaotic and then modify the system to make it stochastic.

The analogy between complexity in chaos and that in stochastic multistability is not meant to be carried too far. Chaos was only used as an introduction to a complexity measure which characterizes program execution time. Chaotic motion, for example, does not settle down. The solution to a master equation, however, typically approaches a steady solution. In a space of unlimited extent, however, containing always more remote states of local stability, distribution function changes *can* continue indefinitely. Chaos in deterministic systems can arise in a simple system described by a few parameters. Our statistical complexities, on the other hand, arise from the existence of a great many competing states of local stability, i.e., from intrinsically complex dynamics.

This discussion of complexity in the presence of multistability is a first rough thrust in a direction which demands more formal skills than this author can supply.

7. CONCLUSION

I have elaborated on the views of refs. 10 and 12. The loop that I have described may not be all that easily realizable in a simple physical form, but it is a model of the circulatory effects that can be found in many-dimensional systems and illustrated in Fig. 1.

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This paper represents the culmination of a journey which started with ref. 6. That work was heavily based on Nico van Kampen's work on fluctuations in vacuum diodes. That reference cited van Kampen's work but did not adequately acknowledge the depth of the relationship. My subsequent evolving thoughts in the area eventually triggered ref. 10 by van Kampen. This present item is my own version of refs. 10 and 12 and resulted largely from extended correspondence and conversation with van Kampen. We seem to agree on all the principal points, but have not come into equilibrium on all the subsidiary questions. Nico, above all others, will understand that the truth in science is uncovered only with hard work, and sometimes only after debate.

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